Large and Moderate Deviation Principles for path-distribution dependent SDEs

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Outline

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- Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq}, \mathbb{P})$ be a probability space and $\{W_t\}$ be an $\mathbb{R}^d$-valued standard Brownian motion.

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Consider

$$\mathrm{d}X^\epsilon(t) = b(t, X^\epsilon(t), \mathscr{L}_{X^\epsilon(t)})\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^\epsilon(t), \mathscr{L}_{X^\epsilon(t)})\mathrm{d}W(t),\ \ t \in [0, T]$$

with initial data $X^\epsilon(0) = x$.

Introduction

Under appropriate assumptions, as $\epsilon \to 0$, X^{ϵ} will tend to the solution of the following deterministic equation:

$$\begin{cases} \mathrm{d}X^0(t) = b(t, X^0(t), \delta_{X^0(t)})\mathrm{d}t, \ \ t \in [0, T], \\ X^0(0) = x, \end{cases} \tag{1}$$

where $\delta_{X^0(t)}$ is a dirac measure at $X^0(t)$.

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- Large deviation principle(LDP): $\lambda(\epsilon) = \frac{1}{\sqrt{\epsilon}}$;
- Central limit theorem(CLT): $\lambda(\epsilon) = 1$;
- Moderate deviation principle(MDP):

$$\lambda(\epsilon) \to +\infty, \quad \sqrt{\epsilon}\lambda(\epsilon) \to 0 \text{ as } \epsilon \to 0.$$

Assume $\mathscr{E}$ is a Polish space and $\mathscr{B}(\mathscr{E})$ is the Borel $\sigma-$algebra. $\{Z^\epsilon\}_{\epsilon>0}$ is a family of $\mathscr{E}-$valued random variables.

Definition (Rate function)

A function $I : \mathscr{E} \to [0, \infty]$ is called a rate function on $\mathscr{E}$, if for each $\alpha \in [0, \infty)$, the level set $\{x \in \mathscr{E} : I(x) \leq \alpha\}$ is a compact subset of $\mathscr{E}$.

Definition (Large deviation principle)

Let I be a rate function on $\mathscr{E}$. The family $\{Z^\epsilon\}_{\epsilon>0}$ is said to satisfy a large deviation principle on $\mathscr{E}$ with rate function I, if the following two conditions are satisfied:

1. (Upper bound) For every closed subset C of $\mathscr{E}$,

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(Z^\epsilon \in C\right) \leq -\inf_{x\in C} I(x).$$

2. (Lower bound) For every open subset O of $\mathscr{E}$,

$$\liminf_{\epsilon\to 0} \epsilon \log \mathbb{P}\left(Z^\epsilon \in O\right) \geq -\inf_{x\in O} I(x).$$

Introduction

LDP for mean-field interacting particle systems:

(1) S. Feng, Large deviations for empirical process of mean-field interacting particle system with unbounded jumps. *Ann. Probab.*, 22(1994), no.4, 2122-2151.

(2) S. Feng, Large deviations for Markov processes with mean field interaction and unbounded jumps. *Probab. Theory Related Fields*, 100(1994), no. 2, 227-252.

(3) W. Liu and L. Wu, Large deviations for empirical measures of mean-field gibbs measures. *Stoch. Proc. Appl.*, 130 (2020), 503-520.

LDP and MDP on $(C([0,T];\mathbb{R}^d), \|\cdot\|_\infty)$:

(1) G. Dos Reis, W. Salkeld, and J. Tugaut, Freidlin-Wentzell LDPs in path space for McKean-Vlasov equations and the functional iterated logarithm law. *Ann. Appl. Probab.*, **29**(2019), 1487-1540.

(2) Y. Suo and C. Yuan, CLT and MDP for McKean-Vlasov SDEs. *Acta Appl. Math.*, 175(2021), Paper No.16.

(3) D. Adams, G.D. Reis, R. Ravaille, W. Salkeld and J. Tugaut, Large Deviations and Exit-times for reflected McKean-Vlasov equations with self-stabilizing terms and superlinear drifts. *Stoch. Proc. Appl.*, 146(2022), 264-310.

Known Results

Idea of the existing results: Recall

$$\mathrm{d}X^{\epsilon}(t) = b(t, X^{\epsilon}(t), \mathscr{L}_{X^{\epsilon}(t)})\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^{\epsilon}(t), \mathscr{L}_{X^{\epsilon}(t)})\mathrm{d}W(t),\ \ t \in [0, T],$$

and

$$\mathrm{d}X^{0}(t) = b(t, X^{0}(t), \delta_{X^{0}(t)})\mathrm{d}t,\ \ t \in [0, T],$$

with initial data $X^{\epsilon}(0) = X^{0}(0) = x$.

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Two steps:

- Step 1: LDPs for Z^ϵ as the parameter ϵ tends to 0,
- Step 2: X^ϵ and Z^ϵ are exponentially equivalent as ϵ goes to 0, i.e. for any $\delta > 0$,

$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}(\|X^\epsilon - Z^\epsilon\|_\infty \geq \delta) = -\infty.$$

Known Results

(4) W. Liu, Y. Song, J. Zhai, T. Zhang: Large and moderate deviation principles for McKean-Vlasov SDEs with jumps, to appear in *Potential Analysis*

(5) W. Hong, S. Li, and W. Liu: Large deviation principle for McKean-Vlasov quasilinear stochastic evolution equations. *Appl. Math. Optim.*, 84(2021), no. 1, suppl., S1119-S1147.

Known Results

The key step is to identify the correct controlled equation.
Assume that there is a unique strong solution X^ϵ.

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with$X^\epsilon(0) = x$.
Then, there exists a measurable map Γ^ϵ such that the solution X^ϵ can be represented as

$$X^\epsilon = \Gamma^\epsilon(W(\cdot)).$$

$X^{\epsilon,h^\epsilon} := \Gamma^\epsilon\left(W(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^\cdot u^\epsilon(s)ds\right)$ is the solution to the following controlled SDE:

$$\begin{aligned}\mathrm{d}X^{\epsilon,u^\epsilon}(t) &= b(t, X^{\epsilon,u^\epsilon}(t), \mathscr{L}_{X^\epsilon(t)})\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X^{\epsilon,u^\epsilon}(t), \mathscr{L}_{X^\epsilon(t)})\mathrm{d}W(t) \\ &\quad + \sigma(t, X^{\epsilon,u^\epsilon}(t), \mathscr{L}_{X^\epsilon(t)})u^\epsilon(t)\mathrm{d}t, \quad t \in [0, T],\end{aligned}$$

where $\mathscr{L}_{X^\epsilon(t)}$ is the distribution of $X^\epsilon(t)$, but not the one of $X^{\epsilon,u^\epsilon}(t)$.

Introduction

A. Matoussi, W. Sabbagh, T. Zhang, Large deviation principle of obstacle problems for Quasilinear Stochastic PDEs. *Appl. Math. Optim.*, 83(2021), no.2, 849-879.

(LDP1) For any $m \in (0, \infty)$, any family $\{u_\epsilon, \epsilon > 0\} \subset \mathcal{S}^m$, and any $\delta > 0$,

$$\lim_{\epsilon \to 0} \mathbb{P}\Big(\|X^{\epsilon, u_\epsilon} - \Gamma^0(u_\epsilon)\|_\infty > \delta \Big) = 0.$$

(LDP2) For any $m \in (0, \infty)$ and any family $\{u_n \in S^m, \ n \in \mathbb{N}\}$ satisfying that u_n converges to some element u in S^m as $n \to \infty$, $\Gamma^0(u_n)$ converges to $\Gamma^0(u)$ in the space $C([0, T]; \mathbb{R}^d)$.

Introduction

LDP on $(C^\alpha([0,T];\mathbb{R}^d), \|\cdot\|_\alpha)$:

1 P. Baldi, G. Ben Arous, and G. Kerkyacharian, Large deviations and the Strassen theorem in Hölder norm. *Stochastic Process. Appl.* 42(1992), no.1, 171-180.

2 G. Ben Arous and M. Ledoux, Grandes déviations de Freidlin-Wentzell en norme Hölderienne. In *Séminaire de Probabilités, XXVIII. Lecture Notes in Math.*, 1583(1994), 293-299. Springer, Berlin.

3 Y.-J. Hu, A large deviation principle for small perturbations of random evolution equations in Hölder norm. *Stochastic Process. Appl.* 68(1997), 83-99.

4 F. Gao, Large deviations for diffusion processes in Hölder norm. *Adv. in Math. (China)*, 26(1997), no.2, 147-158.

5 G. Dos Reis, W. Salkeld, and J. Tugaut, Freidlin-Wentzell LDPs in path space for McKean-Vlasov equations and the functional iterated logarithm law. *Ann. Appl. Probab.*, **29**(2019), 1487-1540.

Known Results

Idea of the existing results: Two steps to establish LDP on Hölder space:

- Step 1: prove LDPs on $C([0,T];\mathbb{R}^d)$.
- Step 2: transfer LDP results from supremum norms to Hölder norms. To derive the following inequality: for each $R > 0$ and $\rho > 0$, there exists $\delta > 0$ and for $v > 0$ such that for $\epsilon \in (0, v)$,

$$\mathbb{P}\left(\|X^\epsilon - Y^u\|_\alpha \geq \rho, \|\sqrt{\epsilon}W - \int_0^\cdot u(s)\mathrm{d}s\|_\infty \leq \delta\right) \lesssim \exp\left(-\frac{R}{\epsilon}\right).$$

Outline

Main Results

- For any fixed $\tau > 0$, denote by $\mathscr{C} := C([-\tau, 0], \mathbb{R}^d)$ the space of all continuous $\mathbb{R}^d-$valued functions defined on $[-\tau, 0]$. It is equipped with the uniform norm

$$\|\xi\|_{\mathscr{C}} := \sup_{\theta \in [-\tau, 0]} |\xi(\theta)|.$$

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$$\|\xi\|_{\mathscr{C}} := \sup_{\theta \in [-\tau, 0]} |\xi(\theta)|.$$

- Let $\mathscr{P}_2^{\mathscr{C}}$ be the collection of all probability measures with finite second moments on $\mathscr{C}$. It is equipped with

$$\mathbb{W}_2(\mu, \nu) := \inf_{\pi \in \Gamma(\mu, \nu)} \left(\int_{\mathscr{C} \times \mathscr{C}} \|\xi - \eta\|_{\mathscr{C}}^2 \pi(\mathrm{d}\xi, \mathrm{d}\eta) \right)^{\frac{1}{2}},$$

where $\Gamma(\mu, \nu)$ denotes the set of all couplings for μ and ν.

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where $\Gamma(\mu, \nu)$ denotes the set of all couplings for μ and ν.

- For $f \in C([-\tau, T], \mathbb{R}^d)$, the $\mathscr{C}$-valued function $\{f_t\}_{t \in [0, T]}$ defined by

$$f_t(\theta) = f(t + \theta), \quad \theta \in [-\tau, 0]$$

is called the segment (or window) process of $\{f(t)\}_{t \in [-\tau, T]}$.

Main Results

Path-distribution dependent SDEs:

Let $b : [0, +\infty) \times \mathscr{C} \times \mathscr{P}_2^{\mathscr{C}} \to \mathbb{R}^d$ and $\sigma : [0, +\infty) \times \mathscr{C} \times \mathscr{P}_2^{\mathscr{C}} \to \mathbb{R}^d \otimes \mathbb{R}^d$ be measurable functions.

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- Consider the following path-distribution dependent SDEs:

$$\mathrm{d}X(t) = b(t, X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(t, X_t, \mathscr{L}_{X_t})\mathrm{d}W(t), \quad X_0 = \xi, \tag{2}$$

  where ξ is an element of $\mathscr{C}$, X_t is the segment process and $\mathscr{L}_{X_t}$ stands for the distribution of X_t.

- Consider

$$\mathrm{d}X^\epsilon(t) = b(t, X_t^\epsilon, \mathscr{L}_{X_t^\epsilon})\mathrm{d}t + \sqrt{\epsilon}\sigma(t, X_t^\epsilon, \mathscr{L}_{X_t^\epsilon})\mathrm{d}W(t), \quad X_0 = \xi \in \mathscr{C}.$$

- Let $\{X^0(t)\}_{t\in[-\tau,T]}$ be the solution of

$$\mathrm{d}X^0(t) = b(t, X_t^0, \delta_{X_t^0})\mathrm{d}t, \quad X_0 = \xi \in \mathscr{C}.$$

Main Results

Assumptions:

(H1) (Continuity) For each $t \geq 0, b(t,\cdot,\cdot)$ is continuous on $\mathscr{C} \times \mathscr{P}_2^{\mathscr{C}}$, and there exists $L > 0$ such that for $t \geq 0$ $\xi, \eta \in \mathscr{C}$, $\mu, \nu \in \mathscr{P}_2^{\mathscr{C}}$,

$$|b(t,\xi,\mu) - b(t,\eta,\nu)|^2 + \|\sigma(t,\xi,\mu) - \sigma(t,\eta,\nu)\|^2 \leq L\left(\|\xi - \eta\|_{\mathscr{C}}^2 + \mathbb{W}_2^2(\mu,\nu)\right).$$

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(H2) (Growth) b is bounded on bounded sets in $[0,\infty) \times \mathscr{C} \times \mathscr{P}_2^{\mathscr{C}}$, and there exists $K > 0$ such that

$$|b(t,0,\mu)|^2 + \|\sigma(t,0,\mu)\|^2 \leq K \left(1 + \mu(\|\cdot\|_{\mathscr{C}}^2)\right), t \geq 0, \mu \in \mathscr{P}_2^{\mathscr{C}}.$$

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(H3) For any $t \in [0,T]$ and $\mu \in \mathscr{P}_2$, $b(t,\cdot,\mu) : \mathscr{C} \to \mathbb{R}^d$ is Frechet differentiable. There exists $L' > 0$ such that

$$\|Db(t,x,\mu) - Db(t,x',\mu)\|_{\mathscr{C}^*} \leq L'\|x-x'\|_{\mathscr{C}}, \forall x,x' \in \mathscr{C}, t \in [0,T], \mu \in \mathscr{P}_2$$

and $\int_0^T \|Db(t, X_t^0, \delta_{X_t^0})\|_{\mathscr{C}^*} \mathrm{d}t < \infty$.

Main Results

Definition

A continuous adapted process $\{X_t\}$ on $\mathscr{C}$ is called a strong solution of (2), if

$$\int_0^t \mathbb{E}|b(s, X_s, \mathscr{L}_{X_s})|\mathrm{d}s + \int_0^t \mathbb{E}\|\sigma(s, X_s, \mathscr{L}_{X_s})\|^2 \mathrm{d}s < \infty, \ t \geq 0$$

and $X(t) := X_t(0)$ satisfies

$$X(t) = \xi(0) + \int_0^t b(s, X_s, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t \sigma(s, X_s, \mathscr{L}_{X_s})\mathrm{d}W_s, \quad t \geq 0.$$

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Existence and Uniqueness of the Solution: According to Theorem 3.1 in [1], under (H1) and (H2) there is a unique strong solution to (2).

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1. X. Huang, M. Röckner, F.-Y. Wang, Nonlinear Fokker-Planck equations for probability measures on path space and path-distribution dependent SDEs, *Discrete Contin. Dyn. Syst.*, 39(2019), 3017-3035.

Main Results

For $u \in L^2([0,T],\mathbb{R}^d)$, let $\{Y^u(t)\}_{t\in[-\tau,T]}$ be the solution of

$$\begin{cases} \frac{\mathrm{d}Y^u(t)}{\mathrm{d}t} = b(t, Y^u_t, \delta_{X^0_t}) + \sigma(t, Y^u_t, \delta_{X^0_t})u(t), \quad t \in (0,T], \\ Y^u_0(t) = \xi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{3}$$

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Theorem 1[Gu, S. 2022]

Assume (H1) and (H2) hold. Then $\{X^\epsilon(\cdot)\}_{\epsilon>0}$ satisfies a LDP on $C([-\tau,T];\mathbb{R}^d)$ with the speed ϵ and the rate function I given by

$$I(g) = \inf_{\{u\in L^2([0,T],\mathbb{R}^d):g=Y^u\}} \left\{ \frac{1}{2}\int_0^T |u(s)|^2 \mathrm{d}s \right\}, \quad g \in C([-\tau,T],\mathbb{R}^d).$$

Assumptions

Let $a(\epsilon) > 0, \epsilon \in (0,1)$ satisfy

$$a(\epsilon) \to +\infty, \quad \frac{\epsilon}{a(\epsilon)^2} \to 0, \quad \epsilon \to 0.$$

Define

$$M^\epsilon(t) := \frac{1}{a(\epsilon)} \left(X^\epsilon(t) - X^0(t) \right), \quad t \in [-\tau, T]$$

and for $t \in [0, T]$

$$M^\epsilon_t(r) := M^\epsilon(t + r), r \in [-\tau, 0].$$

Main Result

For $u \in S$, let K^u be the solution of

$$\begin{cases} \mathrm{d}K^u(t) =_{\mathscr{C}^*} \langle Db(t, X_t^0, \delta_{X_t^0}), K_t^u \rangle_{\mathscr{C}} \mathrm{d}t + \sigma(t, X_t^0, \delta_{X_t^0}) u(t) \mathrm{d}t, \\ K_0^u(t) = 0, \quad t \in [-\tau, 0]. \end{cases}$$

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Theorem 2[Gu, S. 2022]

Assume (H1), (H2) and (H3) hold. Then $\{M^\epsilon(\cdot), \epsilon > 0\}$ satisfies a LDP on $C([-\tau, T], \mathbb{R}^d)$ with speed $\epsilon/a(\epsilon)$ and the rate function I given by

$$I(g) = \inf_{\{u \in L^2([0,T],\mathbb{R}^d): g = K^u\}} \left\{ \frac{1}{2}\int_0^T |u(s)|^2 \mathrm{d}s \right\}, \quad g \in C([-\tau, T], \mathbb{R}^d).$$

Main Result

(H4) σ is bounded and there exists $\beta \in (0, 1]$ such that for each $y \in \mathscr{C}$ and $\mu \in \mathscr{P}^{\mathscr{C}}$, $b(\cdot, y, \mu)$ and $\sigma(\cdot, y, \mu)$ are β-Hölder continuous.

Main Result

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For a Borel set A of Hölder space $C^\alpha([-\tau,1];\mathbb{R}^d)$, denote

$$I(A) := \inf\left\{\frac{1}{2}\int_0^1 |u(s)|^2 \mathrm{d}s : Y^u \in A\right\}.$$

Theorem 3[S. 2022]

Assume (H1), (H2) and (H4) hold. Let $\alpha \in (0, \frac{1}{2})$ and $\xi \in C^\alpha([-\tau,0];\mathbb{R}^d)$. Then for each Borel set A of $C^\alpha([-\tau,1];\mathbb{R}^d)$,

$$-I(\mathring{A}) \le \liminf_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in A) \le \limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in A) \le -I(\overline{A}),$$

where $\mathring{A}$ and $\overline{A}$ are the interior and closure of A in $C^\alpha([-\tau,1];\mathbb{R}^d)$.

Main Result

Proposition 4

Under the conditions of Theorem 3, for each $R > 0$ and $\rho > 0$, there exists $\delta > 0$ and for $v > 0$ such that for $\epsilon \in (0, v)$,

$$\mathbb{P}\left(\|X^\epsilon - Y^u\|_\alpha \geq \rho, \|\sqrt{\epsilon}W - \int_0^\cdot u(s)\mathrm{d}s\|_\infty \leq \delta\right) \lesssim \exp\left(-\frac{R}{\epsilon}\right).$$

Main Result

Let n be an integer and $\{\bar{W}(t)\}_{t\in[0,1]}$ be a n-dimensional Brownian motion.

Lemma[G. Ben Arous, M. Ledoux 1994]

For $\alpha \in (0, \frac{1}{2})$, there exists a constant $C > 0$ independent of n such that for $u, v > 0$,

$$\mathbb{P}\left(\|\bar{W}\|_\alpha \geq u, \|\bar{W}\|_\infty \leq v\right) \leq C \max\left\{1, (\frac{u}{v})^{\frac{1}{\alpha}}\right\} \exp\left\{ -\frac{1}{C}\frac{u^{\frac{1}{\alpha}}}{v^{\frac{1}{\alpha}-2}}\right\}.$$

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Lemma[G. Ben Arous, M. Ledoux 1994]

For $\alpha \in (0, \frac{1}{2})$, there is a constant $C' > 0$ such that for $u > 0$ and $K \in C([0,1])$ with $\|K\|_\infty \leq 1$,

$$\mathbb{P}\left(\|\int_0^\cdot K(s)\mathrm{d}W(s)\|_\alpha \geq u, \|K\|_\infty \leq 1\right) \leq C' \exp\left\{-\frac{u^2}{C'}\right\}.$$

Thanks for Your Attention.